Analysis 1 - Mid-Sem exam - 2008-09

B.Math (Hons.)

Problem 1. *Consider the sequences* $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ *where*

$$a_n = 1 + \left(-\frac{1}{5}\right)^n; \quad b_n = (-1)^n + \frac{2}{n}; \quad c_n = \frac{6n+4}{7n-5}$$

Compute the limit superior and limit inferior (as n tends to infinity) for these sequences.

Solution. **(a)** $a_n = 1 + \left(-\frac{1}{5}\right)^n$

Note that

$$|a_n - 1| = \left(\frac{1}{5}\right)^n \to 0$$

as $n \to \infty$. Hence $a_n \to 1$. Therefore $\limsup a_n = \liminf a_n = 1$.

(b) $b_n = (-1)^n + \frac{2}{n}$

Let $x_n = \sup\limits_{k\geq n} b_k$. Then we have

$$x_n = \begin{cases} 1 + 2/n & \text{if } n \text{ is even} \\ 1 + 2/(n+1) & \text{if } n \text{ is odd} \end{cases}$$

Therefore $\limsup b_n = \lim x_n = 1$.

Let $y_n = \inf\limits_{k\geq n} b_k = -1$. Thus we have $\liminf b_n = \lim y_n = -1$.

(c) $c_n = \frac{6n+4}{7n-5}$

Note that

$$\left|\frac{6n+4}{7n-5} - \frac{6}{7}\right| = \left|\left(\frac{-12}{49n+35}\right)\right| \to 0$$

as $n \to \infty$. Hence $c_n \to 6/7$. Therefore $\limsup c_n = \liminf c_n = 6/7$. □

Problem 2. *Let* $k : [0,1] \to \mathbb{R}$ *be a continuous function. Suppose that k has local maximum at two distinct points* x_1, x_2 *in* $[0,1]$. *Show that k has a local minimum at some point* x_3 *in* $[0,1]$.

Solution. Let $A = [0,1]$. It is enough to show that $k(A)$ is a compact set in $\mathbb{R}$.
Let $\{A_\alpha | \alpha \in I\}$ be an open cover of $k(A)$ for some index set I. Since k is a continuous function, $\{k^{-1}(A_\alpha) | \alpha \in I\}$ is an open cover of A.

Now, as A is a compact set, there exists $\alpha_1, \cdots, \alpha_n \in I$ such that $\bigcup\limits_{i=1}^{n}\{k^{-1}(A_{\alpha_i})\} \supseteq A$. Therefore

$$\bigcup_{i=1}^{n}\{k^{-1}(A_{\alpha_i})\} = k^{-1}(\bigcup_{i=1}^{n}(A_{\alpha_i})) \supseteq A \tag{1}$$

Therefore, $\bigcup_{i=1}^{n} A_{\alpha_i} \supseteq k(A)$. Hence $k(A)$ is compact and hence closed and attains its minimum in particular. □

Problem 3. *Let $\{f_n\}_{n\geq 1}$ be a sequence of real valued continuous functions on $[0,1]$ converging pointwise to a continuous function $f : [0,1] \to \mathbb{R}$. Show that the convergence is uniform if*

$$f_n(x) \geq f_{n+1}(x) \quad \forall x \in [0,1]$$

for all $n \geq 1$. (Hint : Use compactness of $[0,1]$.)

Solution. Firstly observe that, since $f_n(x)$ is decreasing and $f(x) = \inf_n f_n(x)$,

$$|f_n(x) - f(x)| < |f_N(x) - f(x)| \tag{2}$$

for all $n \geq N$.

Let $\epsilon > 0$. Let $x \in [0,1]$. Then, there exists an N_x such that $|f_{N_x}(x) - f(x)| < \epsilon$. Since $f_{N_x} - f$ is continuous on $[0,1]$, there exists a $\delta_{N_x} > 0$, such that

$$|f_{N_x}(y) - f(y)| < \epsilon \quad \forall y \in B(x, \delta_{N_x})$$

Where $B(x,\delta)$ is the open ball centered at x and radius δ. Now, we have that,

$$\bigcup_{x\in[0,1]} B(x, \delta_{N_x}) = [0,1]$$

Since $[0,1]$ is compact, there exists $x_i \in [0,1]$, $i \in \{1, \cdots, n\}$ such that

$$\bigcup_{i=1}^{n} B(x_i, \delta_{N_{x_i}}) = [0,1]$$

Let $N = \max_{1\leq i\leq n} \{N_{x_i}\}$. Then, $|f_N(x) - f(x)| < \epsilon$ for all x. (because of (2)).

Now, again because of (2)

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N \; \forall x$$

Hence f_n converges uniformly. □

Problem 4. *Let D be a nonempty subset of $\mathbb{R}$ and let $h : D \to \mathbb{R}$ be uniformly continuous. If D is bounded show that h is bounded. Use this result to show that $g : (0,\infty) \to \mathbb{R}$ defined by $g(x) = 1/x$ is not uniformly continuous.*

Solution. **Claim:** If $f : D \to \mathbb{R}$ is uniformly continuous, and if $\{x_n\} \subset D$ is cauchy, then $f(x_n)$ is also cauchy. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(y) - f(x)| < \epsilon$. There exists an N for this δ such that $|x_n - x_m| < \delta$ for every $n, m > N$.

Therefore $|f(x_n) - f(x_m)| < \epsilon$, for every $m, n \geq N$.

If h is unbounded on D, there exists a sequence $x_n \in D$ such that $h(x_n)$ is unbounded and $|h(x_n)| > n$. But since x_n is a bounded sequence in $\mathbb{R}$, it has a converging subsequence x_{n_k}. Therefore x_{n_k} is cauchy. But since $|h(x_{n_k})| > n_k$, we get a contradiction, due to the claim above.

Hence h is bounded.

Now, consider g restricted to $(0,1)$. Note that this is unbounded, while the set $(0,1)$ is bounded. Hence g is not uniformly continuous.

□

Problem 5. *Let $u : [0,1] \to \mathbb{R}$ be a continuous function. Define $v : [0,1] \to \mathbb{R}$ by*

$$v(x) = \sup\{u(y) : 0 \leq y \leq x\}$$

Show that v is a continuous function.

Solution. Let $c \in [0,1]$. We shall consider the following cases

1. Suppose $u(c) = v(c)$.
   Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\mid x - c \mid < \delta$ implies $\mid u(x) - u(c) \mid < \epsilon$. Pick c_1 and c_2 such that $max(c - \delta, 0) \leq c_1 \leq c \leq c_2 \leq min(1, c + \delta)$. Note that $u(x) \leq v(c)$ for $x < c$ and $u(x) < u(c) + \epsilon$ for $c \leq x < c_2$. Thus for $x \in (c_1, c_2)$
   $$v(c) - \epsilon = u(c) - \epsilon < u(c_1) \leq v(x) \leq max(v(c), u(c) + \epsilon) = v(c) + \epsilon \tag{3}$$
   From the above equation, it follows that $\mid v(x) - v(c) \mid < \epsilon$ for $x \in (c_1, c_2)$.

2. Suppose $u(c) \neq v(c)$.
   Let $\epsilon = v(c) - u(c)$. $\epsilon > 0$ from the definition of v. Choose $\delta > 0$ such that $\mid x - c \mid < \delta$ implies $\mid u(x) - u(c) \mid < \epsilon$. Then either $u(x) > v(c)$ for all $x \in (c - \delta, c + \delta) \cap [0,1]$, or $u(x) < v(c)$ for all $x \in (c - \delta, c + \delta) \cap [0,1]$. The former is impossible from the definition of v and hence the latter holds which implies that v is constant on $(c - \delta, c + \delta) \cap [0,1]$.

Hence v is continuous. □

Problem 6. *State and prove the mean value theorem.*

Solution. If f is a real continuous function on $[a,b]$ which is differentiable on (a,b) then there exists a point $x \in (a,b)$ at which $f(b) - f(a) = (b-a)f'(x)$.

Let $h(t) = (f(b) - f(a))t - (b-a)f(t)$, where $a \leq t \leq b$. Then h is continuous on $[a,b]$, differentiable on (a,b) and
$$h(a) = f(b)a - f(a)b = h(b) \tag{4}$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a,b)$.

If h is constant, this holds for every x in (a,b).

If $h(t) > h(a)$ for some $t \in (a,b)$, let $x \in [a,b]$ at which h attains its maximum.

From equation (4) we have $x \in (a,b)$. Since h attains maximum at x, $h'(x) = 0$.

If $h(t) < h(a)$ for some $t \in (a,b)$, the same argument applies if we choose for x a point on $[a,b]$ where h attains its minimum.

□

Problem 7. *Show that every bounded sequence of complex numbers has a convergent subsequence.*

Solution. <u>Claim:</u> Every bounded sequence of real numbers has a convergent subsequence

<u>Proof:</u> Let $\{w_n\}$ be a bounded sequence of real numbers, then there exists $[a_1, b_1]$ such that $a_1 \leq w_n \leq b_1 \forall n$. Either $[a_1, (a_1 + b_1)/2]$ or $[(a_1 + b_1)/2, b_1]$ contains infinitely many terms of the sequence $\{w_n\}$. If $[a_1, (a_1 + b_1)/2]$ contains infinitely many terms, let $[a_2, b_2] = [a_1, (a_1 + b_1)/2]$, otherwise, let $[a_2, b_2] = [(a_1 + b_1)/2, b_1]$. By mathematical induction, we can continue this construction and obtain a sequence of intervals $[a_n, b_n]$ such that

1. for each n, $[a_n, b_n]$ contains infinitely many terms of the sequence $\{w_n\}$
2. for each n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
3. for each n, $b_{n+1} - a_{n+1} = (b_n - a_n)/2$

The nested interval theorem implies that the intersection of all those intervals is a singleton set w. We shall now construct a subsequence of $\{w_n\}$ which will converge to w.
Since $[a_1, b_1]$ contains infinitely many terms of $\{w_n\}$, there exists a k_1 such that $w_{k_1} \in [a_1, b_1]$. Since $[a_2, b_2]$ contains infinitely many terms of $\{w_n\}$, there exists $k_2 > k_1$ such that $w_{k_2} \in [a_2, b_2]$. Continuing this process by induction, we obtain a sequence $\{w_{k_n}\} \in [a_n, b_n]$ for each n. $\{w_{k_n}\}$ is a subsequence of $\{w_n\}$ since $k_{n+1} > k_n$ for each n. Since $a_n \to w$, $b_n \to w$ and $a_n \leq w_{k_n} \leq b_n$ for each n, the squeeze theorem implies that $w_{k_n} \to w$ ■

Let $\{z_n\}$ be a sequence of complex numbers, then $z_n = x_n + iy_n$ for some $x_n, y_n \in \mathbb{R}$.

Since z_n is bounded, x_n and y_n are also bounded. So there exists a convergent sub-sequence $\{x_{n_k}\}$. since the sub-sequence $\{y_{n_k}\}$ is also bounded, it has a convergent sub-sequence say $\{y_{n_{k_r}}\}$. Since $\{x_{n_{k_r}}\}$ and $\{y_{n_{k_r}}\}$ converge, $\{z_{n_{k_r}}\}$ also converges.

□

Problem 8. *Consider the series* $\sum_{n\geq 1} a_n$ *where*

$$a_n = \begin{cases} \dfrac{1}{n^2} & \textit{if } n \textit{ is odd} \\ \dfrac{1}{n^3} & \textit{if } n \textit{ is even} \end{cases}$$

Show that this series is convergent but the convergence cannot be determined by ratio or root test.

Solution. Since $|a_n| \leq 1/n^2$ and $\sum 1/n^2$ converges, $\sum a_n$ converges.
Ratio test can be used if either of the following hold:

1. $\limsup \mid \dfrac{a_{n+1}}{a_n} \mid < 1$ in which case $\sum a_n$ converges
2. $\mid \dfrac{a_{n+1}}{a_n} \mid \geq 1$ for $n \geq n_0$ for some fixed integer n_0 in which case $\sum a_n$ diverges.

Now we can see that, $\limsup \mid \dfrac{a_{n+1}}{a_n} \mid > 1$ since whenever $n \geq 4$ is even, $\mid \dfrac{a_{n+1}}{a_n} \mid > 1$ hence the first condition does not hold.

Similarly, the second condition fails to hold when n is odd. Hence one cannot use ratio test.

Root test can be used if:

1. $\limsup \mid a_n \mid^{1/n} < 1$ in which case it converges
2. $\limsup \mid a_n \mid^{1/n} > 1$ in which case it diverges

Since, $\lim(\dfrac{1}{n})^{1/n} = 1$, it follows that $\lim(\dfrac{1}{n^2})^{1/n} = 1$ and $\lim(\dfrac{1}{n^3})^{1/n} = 1$. So $\limsup \mid a_n \mid^{1/n} = 1$. Therefore Root test cannot be applied. □